

Circular modes, beam adapters, and their applications in beam optics

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In the optics of charged particle beams, circular transverse modes can be introduced; they provide an adequate basis for rotation-invariant transformations. A group of these transformations is shown to be identical to a group of the canonical angular momentum preserving mappings. These mappings and the circular modes are parametrized similar to the Courant-Snyder forms for the conventional uncoupled, or planar, case. The planar-to-circular and reverse transformers (beam adapters) are introduced in terms of the circular and planar modes; their implementation on the basis of skew quadrupole blocks is described. Various kinds of matching for beams, adapters and solenoids are considered. Applications of the planar-to-circular, circular-to-planar and circular-to-circular transformers are discussed. A range of applications includes round beams at the interaction region of circular colliders, flat beams for linear colliders, relativistic electron cooling, and ionization cooling.

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I. INTRODUCTION

Linear beam optics normally employs transformations, which either do not couple vertical and horizontal degrees of freedom, or the coupling is weak. In the canonical 4D phase space,

$$\mathbf{x} \equiv \begin{pmatrix} X \\ Y \end{pmatrix} \equiv \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}, \quad (1)$$

these uncoupled transformations \mathcal{P} are described by 4D block-diagonal matrices with independent 2D unimodular blocks for the vertical and horizontal subspaces. For these transformations, particle trajectories are conventionally described by means of uncoupled, or planar, modes. In the Courant-Snyder form [1], the four planar basis vectors can be arranged as columns of a 4×4 block-diagonal matrix:

$$\mathbf{V} = \begin{pmatrix} \frac{\sqrt{\beta_x} \cos(\phi_x)}{\sqrt{\beta_x}} & \frac{\sqrt{\beta_x} \sin(\phi_x)}{\sqrt{\beta_x}} & 0 & 0 \\ \frac{-\alpha_x \cos(\phi_x) - \sin(\phi_x)}{\sqrt{\beta_x}} & \frac{-\alpha_x \sin(\phi_x) + \cos(\phi_x)}{\sqrt{\beta_x}} & 0 & 0 \\ 0 & 0 & \sqrt{\beta_y} \cos(\phi_y) & \sqrt{\beta_y} \sin(\phi_y) \\ 0 & 0 & \frac{-\alpha_y \cos(\phi_y) - \sin(\phi_y)}{\sqrt{\beta_y}} & \frac{-\alpha_y \sin(\phi_y) + \cos(\phi_y)}{\sqrt{\beta_y}} \end{pmatrix}. \quad (2)$$

Any initial phase space vector \mathbf{x} can be expanded over the basis (2)

$$\mathbf{x} = \mathbf{V} \mathbf{a} \quad (3)$$

with its amplitudes $\mathbf{a} = (a_1, a_2, a_3, a_4)^T$. Due to the basis

symplecticity, the amplitudes \mathbf{a} can be considered as new canonical variables. Uncoupled mappings change basis parameters $\alpha_{x,y}, \beta_{x,y}$, and $\phi_{x,y}$, while leaving the amplitudes \mathbf{a} constant. The actions and initial phases $J_{x,y}, \chi_{x,y}$ in the 4D phase space \mathbf{x} can be presented in terms of the amplitudes \mathbf{a} as

$$\mathbf{a} = (\sqrt{2J_x} \sin \chi_x, \sqrt{2J_x} \cos \chi_x, \sqrt{2J_y} \sin \chi_y, \sqrt{2J_y} \cos \chi_y), \quad (4)$$

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where the actions are given by the Courant-Snyder invariants

$$J_x = (a_1^2 + a_2^2)/2 = \gamma_x x^2/2 + \alpha_x x p_x + \beta_x p_x^2/2, \quad (5)$$

$$J_y = (a_3^2 + a_4^2)/2 = \gamma_y y^2/2 + \alpha_y y p_y + \beta_y p_y^2/2,$$

with $\gamma_{x,y} = (1 + \alpha_{x,y}^2)/\beta_{x,y}$.

The structure of the planar basis (2) is preserved only by the uncoupled transformations; for general 4D symplectic transformations the proper basis structure is more complicated; various forms were presented in Refs. [3–5].

For some specific coupled transformations, namely, rotation invariant, the basis reduces to a form which is as simple as the planar one (2); below this basis is referred to as “circular.” As far as we know, the circular basis was first introduced in Ref. [5]; it is rederived here in Sec. III.

For charged particle beams, focusing by means of solenoids or round electrostatic lenses and bending by index $n = 1/2$ dipoles gives a continuous, or local-invariant optics, i.e., such that mapping between any two places is (rotatio) invariant. However, mapping between two specific places can be designed as rotation invariant even on a base of such noninvariant elements as quadrupoles and constant-field dipoles [6]; the whole mapping in this case can be referred to as block invariant. Optical schemes with actual local or block invariance are discussed for muon transport [7], circular colliders (see list of references in, e.g. [9]), and relativistic electron cooling [6]. An important property of rotation-invariant mappings is that they preserve the canonical angular momentum (CAM); this and inverse statements are proven in the next section. For invariant transformations, the adequate basis is constructed from circular modes; this is as obvious for charged particle beams as it is for light. A symplectic circular basis, analogous to the Courant-Snyder uncoupled form, is presented in Sec. III.

After both planar and circular polarized modes are introduced, a problem of their mutual transformation can be considered. When both bases are symplectic, they can be mapped onto each other; thus this transformation can always be done. Such an idea was originally proposed by one of the authors (Ya.D.) to reduce the beam-beam effects in circular colliders; he found that an uncoupled beam state can be transformed into a round whirled state and back. He then called these planar-circular transformers “beam adapters” [10]. If one of the emittances of the coming uncoupled beam can be neglected, then the outgoing beam would be of definite-sign spirality, CAM dominated state [6], and its transverse motion could be completely cancelled inside a matched solenoid. This effective elimination of transverse temperature of coming flat beam can be essential for relativistic electron cooling [11,12]. A particular realization of the adapting optics has been found in Ref. [13]. One more application of the adapter was proposed in Ref. [14]: to get a flat electron beam for a linear collider from a round beam emitted by a magnetized cathode. In Ref. [15], a general requirement on the magnetized-to-flat mapping was discussed; the properties of the involved quadrupole blocks were formulated in terms of the Courant-Snyder parameters;

the emittance preservation for both the canonical emittances was shown. Recently, the magnetized-to-flat beam transformation was demonstrated experimentally at Fermilab [16,17]. In the paper below, the beam adapters are considered as planar-circular transformers, which gives a straightforward way to present all their features. Finally, various applications of the circular modes and beam adapters are discussed.

II. ROTATION-INVARIANT TRANSFORMATIONS

A group of rotations in the transverse plane can be presented by matrices

$$\mathcal{R}(\theta) = \begin{pmatrix} \mathbf{c} \mathbf{I} & \mathbf{s} \mathbf{I} \\ -\mathbf{s} \mathbf{I} & \mathbf{c} \mathbf{I} \end{pmatrix}, \quad (6)$$

with $\mathbf{c} = \cos \theta$, $\mathbf{s} = \sin \theta$ and \mathbf{I} is the 2×2 identity matrix. The rotation invariance of a transformation \mathcal{T} means that \mathcal{T} commutes with the rotations:

$$\mathcal{R} \mathcal{T} - \mathcal{T} \mathcal{R} = 0. \quad (7)$$

This condition is equivalent to its particular case of an infinitesimal rotation by an angle $d\theta$,

$$\mathcal{R} = \mathcal{I} + \mathcal{G} d\theta, \quad \mathcal{G} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}, \quad \mathcal{G}^2 = -\mathcal{I}, \quad (8)$$

where \mathcal{I} and \mathbf{I} are 4×4 and 2×2 identity matrices correspondingly. Then, the invariance condition reduces to a commutation of the mapping \mathcal{T} with the infinitesimal operator \mathcal{G}

$$\mathcal{G} \mathcal{T} - \mathcal{T} \mathcal{G} = 0. \quad (9)$$

The mapping symplecticity is conventionally expressed as

$$\mathcal{T}^T \mathcal{S} \mathcal{T} = \mathcal{S}, \quad (10)$$

where

$$\mathcal{S} = \begin{pmatrix} \mathbf{J} & 0 \\ 0 & \mathbf{J} \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{S}^2 = -\mathcal{I}. \quad (11)$$

The matrix \mathcal{S} is usually referred to as the symplectic unit, \mathcal{I} is the 4×4 identity matrix and the superscript T stands for the transposing.

It can be shown now that symplectic invariant transformations \mathcal{T} preserve the canonical angular momentum (CAM)

$$M \equiv x p_y - y p_x \equiv \frac{1}{2} \mathbf{x}^T \mathcal{L} \mathbf{x}, \quad (12)$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & \mathcal{J} \\ -\mathcal{J} & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{L}^2 = \mathcal{I}. \quad (13)$$

Note that the CAM matrix \mathcal{L} is rotation invariant:

$$\mathcal{G} \cdot \mathcal{L} - \mathcal{L} \cdot \mathcal{G} = 0. \quad (14)$$

In terms of its matrix \mathcal{L} , CAM preservation under the mapping \mathcal{T} can be expressed as

$$\mathcal{T}^T \mathcal{L} \mathcal{T} = \mathcal{L}. \quad (15)$$

To prove that this is true when conditions (10) and (9) are provided, it is convenient to use the relation between the infinitesimal operator \mathcal{G} , the symplectic unit matrix \mathcal{S} , and the CAM matrix \mathcal{L} :

$$\mathcal{S} \mathcal{L} = \mathcal{L} \mathcal{S} = -\mathcal{G}, \quad (16)$$

which is straightforward to prove. It means that the matrices \mathcal{S} , \mathcal{L} , and \mathcal{G} form an algebra: any of their products returns one of them. From Eqs. (14) and (16) the symplecticity matrix can be presented as

$$\mathcal{S} = -\mathcal{L} \mathcal{G}. \quad (17)$$

Being substituted in the symplecticity condition (10), after the commutation (9), it leads to the CAM preservation (15). Thus, the invariant transformations preserve the CAM.

Having shown that the mapping invariance leads to the CAM preservation, a reverse statement can be proven as well: if a symplectic mapping preserves the CAM of any initial state, it is rotationally invariant. Indeed, with the matrix \mathcal{T}^T expressed from the symplecticity condition (10) and substituted in the CAM preservation (15), it leads to what can be seen as the invariance property (9) when Eq. (16) is used. Thus, mapping invariance gives rise to CAM preservation and vice versa, so these properties are absolutely equivalent.

A general form of the CAM-preserving matrices was found by Pozdeev [18] and Perevedentsev [19]; in the rest of this section we are following Perevedentsev. The invariance condition (9) applied to the mapping presented in a block 2×2 form

$$\mathcal{T} = \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix} \quad (18)$$

immediately yields $T_{xx} = T_{yy}$ and $T_{xy} = -T_{yx}$ or

$$\mathcal{T} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \quad (19)$$

The symplecticity condition (10) applied to a matrix of such a form (19) results in

$$A^T J A + B^T J B = J, \quad (20)$$

$$A^T J B - B^T J A = 0. \quad (21)$$

For arbitrary matrix A , it is true that $A^T J A = |A| J$; thus, Eq. (20) gives

$$|A| + |B| = 1. \quad (22)$$

The condition (21) presented as

$$J B A^{-1} = (B A^{-1})^T J$$

yields

$$B = A \times \text{const}. \quad (23)$$

It follows from Eqs. (22) and (23) that the matrices A and B can be presented as

$$A = T \cos \theta, \quad B = T \sin \theta, \quad (24)$$

where T is an arbitrary 2×2 matrix with $|T| = 1$ and θ is an arbitrary parameter. Thus, it leads to a conclusion that 4×4 matrices of a form

$$\mathcal{T} = \begin{pmatrix} T \cos \theta & T \sin \theta \\ -T \sin \theta & T \cos \theta \end{pmatrix} \equiv \mathcal{R}(\theta) \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \quad (25)$$

present a group of symplectic rotation invariant mappings identical to the CAM-preserving group of transformations.

One more interesting transformation is a mirror reflection:

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (26)$$

This symplectic transformation does not commute with rotations, so it is not rotation invariant and cannot be implemented by rotation-invariant optics. Combined with the rotation invariant group, it leads to such mappings as

$$\mathcal{T}_- = \mathcal{T} \mathcal{M} = \mathcal{R}(\theta) \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} \quad (27)$$

which invert a sign of the CAM, preserving the CAM absolute value and beam rotation symmetry. In Ref. [6] a generalized Busch's theorem was proven: if a rotation-invariant laminar beam is linearly transformed into a rotation-invariant beam again, then the absolute value of the CAM is preserved for any particle of this beam. Transformations \mathcal{T} (25) and \mathcal{T}_- (27) together form a group of CAM-value preserving mappings. Thus, according to this theorem, they cover all the linear symplectic transformations, which preserve beam rotation symmetry for any initial round beam state. Reflection-like transformations \mathcal{T}_- can be implemented by means of two adapters, discussed in Secs. IV and V.

A parametrization of the 2×2 unimodular matrix T can be taken in the conventional Courant-Snyder form, in terms of its input α_1, β_1 and output α_2, β_2 parameters and a phase advance μ (see, e.g. [2]):

$$\mathbb{T} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos \mu + \alpha_1 \sin \mu) & \sqrt{\beta_1 \beta_2} \sin \mu \\ -\frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \mu + \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \cos \mu & \sqrt{\frac{\beta_1}{\beta_2}}(\cos \mu - \alpha_2 \sin \mu) \end{pmatrix}, \quad (28)$$

where the subscript 1 of the Courant-Snyder parameters relates to an initial state, and 2 to a final state.

III. CIRCULAR BASIS

In this section, the eigenbasis of the rotation-invariant transformations is constructed. First, some heuristic ideas are used for the construction of the matrix, comprised of four basis vectors. Then, free parameters of this form are taken to make this form symplectic. Finally, it is shown how the remaining free parameters are changed under the rotation-invariant transformations (25).

Rotation-invariant transformations preserve the CAM. The simplest vector with a nonzero CAM can be given as

$$\mathbf{u}_0 = (b, 0, 0, p_t)^T \quad (29)$$

with an arbitrary offset b and the tangential momentum p_t . From \mathbf{u}_0 , two orthogonal vectors can be constructed by rotations:

$$\begin{aligned} \mathbf{u}_1 &= (b \cos \phi_+, -p_t \sin \phi_+, b \sin \phi_+, p_t \cos \phi_+)^T, \\ \mathbf{u}_2 &= (b \sin \phi_+, p_t \cos \phi_+, -b \cos \phi_+, p_t \sin \phi_+)^T, \end{aligned} \quad (30)$$

where ϕ_+ is an arbitrary parameter. An additional pair of orthogonal vectors can be constructed from

$$\tilde{\mathbf{u}}_0 = (-b, 0, 0, p_t)^T \quad (31)$$

in the same way:

$$\begin{aligned} \mathbf{u}_3 &= (-b \cos \phi_-, p_t \sin \phi_-, b \sin \phi_-, p_t \cos \phi_-)^T, \\ \mathbf{u}_4 &= (-b \sin \phi_-, -p_t \cos \phi_-, -b \cos \phi_-, p_t \sin \phi_-)^T. \end{aligned} \quad (32)$$

The structure of the four vectors (30) and (32) is preserved by rotations \mathcal{R} , but it is not general enough to be preserved by the rotation-invariant transformations (25). The reason is that these vectors contain only the tangential momentum p_t having a zero normal (radial) component $p_n = (xp_x + yp_y)/\sqrt{x^2 + y^2}$, which is not general enough. With the normal momentum p_n included, the matrix of the vectors $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ changes as follows:

$$\mathbf{U} = \begin{pmatrix} b \cos \phi_+ & b \sin \phi_+ & -b \cos \phi_- & -b \sin \phi_- \\ -p_t \sin \phi_+ + p_n \cos \phi_+ & p_t \cos \phi_+ + p_n \sin \phi_+ & p_t \sin \phi_- - p_n \cos \phi_- & -p_t \cos \phi_- + p_n \sin \phi_- \\ b \sin \phi_+ & -b \cos \phi_+ & b \sin \phi_- & -b \cos \phi_- \\ p_t \cos \phi_+ + p_n \sin \phi_+ & p_t \sin \phi_+ - p_n \cos \phi_+ & p_t \cos \phi_- + p_n \sin \phi_- & p_t \sin \phi_- - p_n \cos \phi_- \end{pmatrix}. \quad (33)$$

To be a valid basis for the rotation-invariant transformations, it is necessary for the set \mathbf{U} to be symplectic. It is straightforward to see that the symplecticity condition (10) is satisfied for the matrix \mathbf{U} if the tangential momentum p_t is in a specific relation with the offset: $p_t = 1/(2b)$. This enables the CAM to have only certain values for the basis vectors: $M = 1/2$ for the first pair \mathbf{u}_1 and \mathbf{u}_2 , and $M = -1/2$ for the second pair \mathbf{u}_3 and \mathbf{u}_4 .

After the symplecticity of the set of vectors (33) is applied, the final remaining point is to find out how it is changed under the invariant transformations (25). Instead of the offset b and the normal momentum p_n , new parameters, β and α , are more convenient to use:

$$b = \sqrt{\beta/2}; p_n = -\alpha/\sqrt{2\beta}. \quad (34)$$

The matrix \mathbf{U} then becomes the following function of its parameters $\alpha, \beta, \phi_+, \phi_-$:

$$\mathbf{U} \equiv \mathbf{U}(\alpha, \beta, \phi_+, \phi_-) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{\beta} \cos \phi_+}{-\sin \phi_+ - \alpha \cos \phi_+} & \frac{\sqrt{\beta} \sin \phi_+}{\cos \phi_+ - \alpha \sin \phi_+} & \frac{-\sqrt{\beta} \cos \phi_-}{\sin \phi_- + \alpha \cos \phi_-} & \frac{-\sqrt{\beta} \sin \phi_-}{-\cos \phi_- + \alpha \sin \phi_-} \\ \frac{\sqrt{\beta} \sin \phi_+}{\cos \phi_+ - \alpha \sin \phi_+} & \frac{-\sqrt{\beta} \cos \phi_+}{\sin \phi_+ + \alpha \cos \phi_+} & \frac{\sqrt{\beta} \sin \phi_-}{\cos \phi_- - \alpha \sin \phi_-} & \frac{-\sqrt{\beta} \cos \phi_-}{\sin \phi_- + \alpha \cos \phi_-} \end{pmatrix}. \quad (35)$$

This circular basis is almost identical to that introduced in Ref. [5]. The only difference is that the parameters β and α in Eq. (35) are by a factor of 2 smaller than Lebedev-Bogacz's parameters. This choice for normalization is used here for a certain compatibility of the form (35) with the rotation-invariant transformations as they are presented in Eq. (28). Indeed, the invariant transformation \mathcal{T} (25) parameterized by block \mathbf{T} (28) can be applied to the set of circular vectors \mathbf{U} (35). Without any loss of generality, the input Courant-Snyder parameters of the mapping can be matched with the vectors: $\alpha_1 = \alpha; \beta_1 = \beta$. After that, the output vector is found as

$$\tilde{\mathbf{U}} \equiv \mathcal{T}\mathbf{U}(\alpha, \beta, \phi_+, \phi_-) = \mathbf{U}(\alpha_2, \beta_2, \phi_+ + \mu - \theta, \phi_- + \mu + \theta). \quad (36)$$

This result completes the basis construction for the rotation-invariant mappings. It shows that the structure of the symplectic set of vectors \mathbf{U} (35) is preserved under these transformations; thus this set forms the eigenbasis of the rotation-invariant mappings. At this mapping, the vectors expand (change their β parameter), acquire some normal momentum (change their α parameter), and turn (change their phases ϕ_+ and ϕ_-). Note that vectors of the same spirality (sign of the CAM) are turned by the same angles. Circular modes for beam optics were considered in Ref. [10] for the description of round beams in the IP region of circular colliders. In a slightly different form than Eq. (35), the circular modes were derived in Ref. [5] as a particular case of the general 4D symplectic eigenvectors presented in that paper.

Having defined the circular Courant-Snyder parameters according to Eq. (36), any phase space vector \mathbf{x} can be expanded over this rotating basis:

$$\mathbf{x} = \mathbf{U} \mathbf{a}. \quad (37)$$

In this presentation, the parameters of the circular basis are changed after the transformation, while the 4D vector of amplitudes $\mathbf{a} = (a_1, a_2, a_3, a_4)^T$ remains constant. Similar to the planar basis (2), relative values of the same-pair amplitudes (same spirality for the circular basis) relate to the phases of the excited modes, while the sums of the same-pair amplitude squared give the corresponding actions. These actions, or Courant-Snyder invariants of the circular modes, can be expressed in terms of the particle coordinates (x, p_x, y, p_y) . Canonically conjugated actions J_{\pm} and phases χ_{\pm} for posi-

tive and negative spirality modes are given by the same canonical transformation as for the planar modes (4):

$$\mathbf{a} = (\sqrt{2J_+} \sin \chi_+, \sqrt{2J_+} \cos \chi_+, \sqrt{2J_-} \sin \chi_-, \sqrt{2J_-} \cos \chi_-). \quad (38)$$

Taking the amplitudes from their definition (37), the actions can be expressed in terms of 2D vectors of the offset and transverse momentum $\vec{r} = (x, y), \vec{p} = (p_x, p_y)$:

$$J_{\pm} = \gamma \vec{r}^2 / 4 + \alpha \vec{r} \cdot \vec{p} / 2 + \beta \vec{p}^2 / 4 \pm M / 2, \quad (39)$$

where $\gamma \equiv (1 + \alpha^2) / \beta$ and $M = xp_y - yp_x$ is the CAM. Note a similarity of this expression to the corresponding formula in the planar case (5).

The preservation of circular actions J_{\pm} under the invariant mappings means that both their sum and difference are preserved as well:

$$J_+ - J_- = M = \text{const},$$

$$J_+ + J_- = \gamma \vec{r}^2 / 2 + \alpha \vec{r} \cdot \vec{p} + \beta \vec{p}^2 / 2 = \text{const}. \quad (40)$$

Inverse expressions are found as

$$\vec{r}^2 = \beta [J_+ + J_- + 2\sqrt{J_+ J_-} \cos \psi]$$

$$\vec{p}^2 = [(J_+ + J_-)(1 + \alpha^2) + 2\sqrt{J_+ J_-}(-1 + \alpha^2) \cos \psi + 4\sqrt{J_+ J_-} \alpha \sin \psi] / \beta \quad (41)$$

$$\vec{r} \cdot \vec{p} = -\alpha (J_+ + J_-) - 2\sqrt{J_+ J_-} \alpha \cos \psi - 2\sqrt{J_+ J_-} \sin \psi,$$

where $\psi = \phi_+ + \chi_+ + \phi_- + \chi_-$. When only one of the two circular modes is excited (either J_+ or J_- is zero), then

$$\vec{r}^2 = \beta J, \quad \vec{p}^2 = \gamma J, \quad \vec{r} \cdot \vec{p} = -\alpha J, \quad M = \pm J. \quad (42)$$

Due to the basis symplecticity, the amplitudes \mathbf{a} can be considered as new canonical coordinates, where a_1 is conjugated with a_2 and a_3 with a_4 . One more useful canonical transformation is given by the circular basis $\mathbf{U}(\alpha, \beta, \phi_+, \phi_-)$ taken for some fixed values of the phases ϕ_+, ϕ_- , say, $\phi_+ = 0, \phi_- = 0$. Let

$$\mathbf{U}_0(\alpha, \beta) \equiv \mathbf{U}(\alpha, \beta, 0, 0) \quad (43)$$

be such a fixed-phase basis; then new canonical coordinates $\tilde{\mathbf{a}} \equiv (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4)$ can be introduced by a symplectic transformation \mathbf{U}_0 as

$$\mathbf{x} = \mathbf{U}_0 \tilde{\mathbf{a}}. \quad (44)$$

These new coordinates

$$\tilde{\mathbf{a}} = \mathbf{U}_0^{-1} \mathbf{x} = \mathbf{U}(\alpha, \beta, 0, 0)^{-1} \mathbf{U}(\alpha, \beta, \phi_+, \phi_-) \tilde{\mathbf{a}}$$

are nothing else but the constant amplitudes, \mathbf{a} , rotated by the phases $\phi_+, -\phi_-$:

$$\begin{aligned} \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} &= \begin{pmatrix} \cos \phi_+ & \sin \phi_+ \\ -\sin \phi_+ & \cos \phi_+ \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \sqrt{2J_+} \begin{pmatrix} \sin(\phi_+ + \chi_+) \\ \cos(\phi_+ + \chi_+) \end{pmatrix}, \\ \begin{pmatrix} \tilde{a}_3 \\ \tilde{a}_4 \end{pmatrix} &= \begin{pmatrix} \cos \phi_- & \sin \phi_- \\ -\sin \phi_- & \cos \phi_- \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} \\ &= \sqrt{2J_-} \begin{pmatrix} \sin(\phi_- + \chi_-) \\ \cos(\phi_- + \chi_-) \end{pmatrix}. \end{aligned} \quad (45)$$

IV. ADAPTERS

Both planar \mathbf{V} (2) and circular \mathbf{U} (35) basic sets are symplectic; therefore, they can be mapped on each other. Symplectic transformations

$$\mathcal{C} = \mathbf{U} \mathbf{V}^{-1} \quad \text{and} \quad \tilde{\mathcal{C}} = \mathbf{V} \mathbf{U}^{-1} \quad (46)$$

map the planar basis \mathbf{V} on the circular basis \mathbf{U} , and back, respectively. Note that the planar-to-circular transformation \mathcal{C} maps the horizontal and vertical phase spaces on the modes of opposite spiralities. The initial state of a particle \mathbf{x} expanded over the planar basis as $\mathbf{x} = \mathbf{V} \cdot \mathbf{a}$ is characterized by a vector of the amplitudes $\mathbf{a} = (a_1, a_2, a_3, a_4)^T$. Then, the planar-to-circular transformation \mathcal{C} converts the initial state \mathbf{x} into a new one

$$\tilde{\mathbf{x}} = \mathcal{C} \mathbf{x} = \mathbf{U} \mathbf{V}^{-1} \mathbf{V} \mathbf{a} = \mathbf{U} \mathbf{a}$$

with the same amplitudes of expansion over the circular basis; the same statement is true for the opposite transformation $\tilde{\mathcal{C}}$. As a consequence, the corresponding planar and circular Courant-Snyder invariants are equal:

$$J_x = J_+, \quad J_y = J_-. \quad (47)$$

Note that every invariant assumes here its own Courant-Snyder parameters: $J_{x,y}$ are calculated with $\alpha_{x,y}, \beta_{x,y}$ of the planar basis \mathbf{V} (2), while J_{\pm} assumes α, β of the circular basis \mathbf{U} (35).

Principal ideas of the planar-to-circular \mathcal{C} or reverse $\tilde{\mathcal{C}}$ mappings were originally proposed by one of the authors for round beam schemes in circular colliders [10] and, later, for

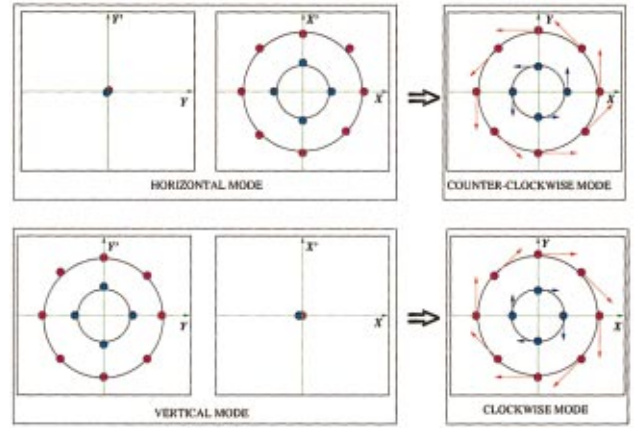


FIG. 1. (Color) Schematic illustration of the planar-to-circular beam adapter: horizontally and vertically polarized modes are transformed into circular modes of opposite spiralities. Blue and red dots represent particles with smaller or larger actions. Arrows on the circular mode portraits show particle momenta, proportional to the offsets. For simplicity, all the phase portraits are depicted as circles; generally, tilted ellipses are mapped onto each other. Direction of external arrows \Rightarrow specify the direction of transformation. Reverse direction of both upper and lower arrows (\Leftarrow) would correspond to the reverse, circular-to-planar adapter.

electron cooling [11]. Optical devices realizing such transformations were named as *beam adapters*, which underlines their shaping role for the beam phase portrait. Adaptive transformations are illustrated schematically by Fig. 1.

For circular colliders, round beams in the interaction region can significantly increase the beam-beam limit of the luminosity [8,9,20]. It can be shown that a proper adapter transforms an incoming uncoupled beam into a rotation-invariant outgoing beam, and the rotation invariance would be guaranteed not only at the interaction point, but in the whole space around it, bounded by the nearest up- and downstream quadrupoles. Indeed, homogeneous distributions over the horizontal and vertical phases for the incoming uncoupled beam turn into a homogeneous distribution over the circular phases in the outgoing beam if the planar-to-circular mapping $\mathcal{C} = \mathbf{U} \mathbf{V}^{-1}$ (46) is matched with the beam, i.e., the Courant-Snyder parameters $\alpha_{x,y}, \beta_{x,y}$ of the mapping are equal to those of the beam. Thus, any matched adapter transforms uncoupled phase-homogeneous beams into rotation-invariant beams. After the interaction region, the round beam can be turned back to a new uncoupled state by means of the reverse transformation. Note that the matched planar-to-circular adapter \mathcal{C} makes outgoing beams round for any ratio of the vertical to horizontal emittances and any machine tunes, contrary to schemes such as those proposed in Ref. [8] and implemented at CESR [20]. Note also that the revolution matrix at the interaction point makes only a transverse turn of the circular basis; this matrix is obviously rotation invariant.

Adapters can also be effectively used for purposes of the relativistic electron cooling, transforming a naturally flat and hot electron beam in a cooling storage ring into a cold elliptical or round beam inside the matched cooling solenoid [11]; the resulting dramatical reduction of the electron temperature

in the cooling section can be crucial for the cooling process. Indeed, when $J_y=0$, only a positive circular mode is excited after mapping \mathcal{C} , making the canonical angular momentum (CAM) a function of the beam offset: $M=r^2/\beta$, according to Eq. (42). Immersing this beam inside the solenoid with the field

$$B=2c/(e\beta) \quad (48)$$

turns the transverse motion to zero; in this matched solenoid, electrons travel strictly along the magnetic field, having zero Larmor radii.

A pair of matched adapters can provide a reflectionlike mapping \mathcal{T}_- , Eq. (27). Indeed, if the first adapter transforms initial circular modes into planar modes, say, $+\Rightarrow x, -\Rightarrow y$, the second can make these planar modes circular again, but with switched spiralities, $x\Rightarrow -, y\Rightarrow +$; thus, eventually the circular modes are transformed as $+\Rightarrow -, -\Rightarrow +$, which means that the sign of the CAM is changed.

V. IMPLEMENTATION OF ADAPTERS

Being symplectic, the adaptive transformations $\mathcal{C}, \bar{\mathcal{C}}$ can be realized. Being linear, they can be realized of quadrupoles. To provide coupling, some quadrupoles must be skew. The question is ‘‘How can it be done?’’ Principle ideas were proposed in Refs. [10,11] and then in more detail in [13]. It was found that particular adaptive transformations can be provided by a skew quadrupole triplet.

The transformation $\mathcal{C}=\mathbf{U}\mathbf{V}^{-1}$ is constructed from given circular and planar bases. Let the circular basis \mathbf{U} be taken for a waist point, where $\alpha=0$, with the phases $\phi_+=-\phi_-=-\pi/4$, while the planar basis is taken with $\alpha_x=\alpha_y=\alpha_0$, $\beta_x=\beta_y=\beta_0$ and $\phi_x=\phi_y=\phi_0$. It is straightforward to show that in this particular case, the adaptive transformation \mathcal{C} reduces to an uncoupled transformation in a frame rotated by $\pi/4$. This can be expressed as

$$\mathcal{C}=\mathcal{R}(\pi/4)\langle\mathbf{M},\mathbf{N}\rangle\mathcal{R}(-\pi/4), \quad (49)$$

where $\langle\mathbf{M},\mathbf{N}\rangle$ stands for a block-diagonal 4×4 matrix with \mathbf{M} and \mathbf{N} as its 2×2 diagonal blocks:

$$\mathbf{M}=\begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\phi_0-\alpha_0\sin\phi_0) & -\sqrt{\beta\beta_0}\sin\phi_0 \\ \frac{\alpha_0\cos\phi_0+\sin\phi_0}{\sqrt{\beta\beta_0}} & \sqrt{\frac{\beta_0}{\beta}}\cos\phi_0 \end{pmatrix} \quad (50)$$

and

$$\mathbf{N}=\begin{pmatrix} -\sqrt{\frac{\beta}{\beta_0}}(\alpha_0\cos\phi_0+\sin\phi_0) & -\sqrt{\beta\beta_0}\cos\phi_0 \\ \frac{\cos\phi_0-\alpha_0\sin\phi_0}{\sqrt{\beta\beta_0}} & -\sqrt{\frac{\beta_0}{\beta}}\sin\phi_0 \end{pmatrix}. \quad (51)$$

For this transformation, the phases ϕ_x, ϕ_y of the initial planar phase space vector and those ϕ_+, ϕ_- of the final circular vector are related as $\phi_+=\phi_x-\phi_0-\pi/4, \phi_-=-\phi_y+\phi_0+\pi/4$. Note that the blocks \mathbf{M} and \mathbf{N} look almost like the standard Courant-Snyder form, Eq. (28). Obviously, these 2×2 matrices are characterized by identical sets of the Courant-Snyder parameters: in terms of Eq. (28), $\alpha_1=\alpha_0, \alpha_2=0$ and $\beta_1=\beta_0, \beta_2=\beta$ for both of them, with the phase advances shifted by $\pi/2$, namely $\mu_M=-\phi_0, \mu_N=-\pi/2-\phi_0$ for \mathbf{M} and \mathbf{N} blocks, respectively. This relation can also be formulated as

$$\mathbf{N}=\mathbf{F}\mathbf{M} \quad \mathbf{F}=\begin{pmatrix} 0 & -\beta \\ 1/\beta & 0 \end{pmatrix}. \quad (52)$$

In other words, this particular adapter can be realized as a sequence of skew quadrupoles, with the condition (52) between the horizontal and vertical matrices in the natural (unrotated) frame of the quadrupoles. This condition on the unimodular 2×2 matrices is equivalent to $2\times 2-1=3$ independent conditions on their elements; thus a skew triplet of quadrupoles with variable gradients can do the job. If the circular β parameter is not fixed, only 2 conditions remain, so 2 variable quadrupoles are sufficient.

It becomes clearer now how an adapting transformation \mathcal{C} can be realized for arbitrary given planar $\alpha_{x,y}, \beta_{x,y}$ and circular α, β Courant-Snyder parameters. First of all, the initial planar basis can be mapped onto another planar basis with identical Courant-Snyder parameters, $\alpha_x=\alpha_y, \beta_x=\beta_y$, which could be done by means of 2 quadrupoles with variable field gradients. Then, the described specific adapter can be applied to this second basis, mapping it onto an unspecified circular basis, which would require 2 more quadrupoles. Finally, a transformation of this unspecified circular basis onto the given circular basis can be provided by 2 quadrupoles upstream from the specific adapter. Thus, $2+2+2=6$ quadrupoles with variable strength can provide the mapping of a given uncoupled beam state onto a given circular state.

For some purposes, it could be useful to have a nonround laminar vortex state. This goal can be reached by applying the planar-to-circular adapter (49) to an initially flat beam where Courant-Snyder parameters differ from the planar parameters of the adapter. In this case, the outgoing beam would have a cross section as a tilted ellipse, where the tilt and aspect ratio would depend on the beam and adapter parameters.

An experimental study of the magnetized-to-flat transformations is currently being done in Fermilab [16,17]. One of the optical schemes used in these experiments is shown in Fig. 2, where the two half-axes and the tilt angle of the beam ellipse are presented along the beam line. The simulations are done with the OptiM code [5]. A finite value of the smaller half-axis at the exit is determined by the beam temperature at the cathode.

VI. CIRCULAR EIGENMODES FOR A SOLENOID

The circular modes described in Sec. III present an adequate basis for any rotation-invariant transformations. The

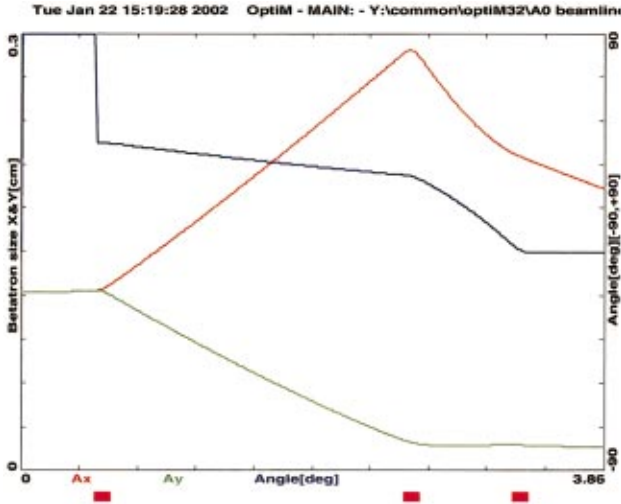


FIG. 2. (Color) Beam ellipse half-axes and tilt angle along a part of the beam line. The beam energy is 16.5 MeV, and the 10.2 cm length skew quads are -30.4 , 52.6 , and -54.9 G/cm of the field gradient. At the cathode, the beam of 0.8 mm radius and 4 eV temperature is immersed in the solenoidal field of 750 G.

choice of the initial Courant-Snyder parameters can be made taking into account the properties of the incoming beam or some ideas related to the convenience or physical sense of the description. In some cases, this choice can be made by the optics itself. In this section, specific circular modes for a solenoid are discussed. Inside an extended solenoid, the modes can be defined in such a way that, while the beam travels along the field, their Courant-Snyder parameters remain constant, and only the phases run. Being rotation-invariant, solenoidal transformation T_s from the entrance to an arbitrary coordinate z inside the solenoid can be presented as Eq. (25):

$$T_s = \mathcal{R}(-\theta_s/2) \langle T_s, T_s \rangle \quad (53)$$

with

$$T_s = \begin{pmatrix} \cos(\theta_s/2) & \beta_s \sin(\theta_s/2) \\ -\beta_s^{-1} \sin(\theta_s/2) & \cos(\theta_s/2) \end{pmatrix}. \quad (54)$$

Here $\theta_s = eBz/(p_0c) \equiv z/\rho$ is the cyclotron phase advance inside the field B for a particle with the longitudinal momentum p_0 . The parameter

$$\beta_s = 2c/(eB) \quad (55)$$

can be referred to as the Larmor β function. From here, it follows that the Courant-Snyder parameters of the circular basis with $\beta = \beta_s$ and $\alpha = 0$ are preserved inside the solenoid: the first pair of the basis vectors turns by an angle $\Delta\phi_+ = \theta_s/2 + \theta_s/2 = \theta_s$ and the second pair by $\Delta\phi_- = -\theta_s/2 + \theta_s/2 = 0$, i.e., remains unturned.

It is straightforward to see that the canonical variables $\tilde{\mathbf{a}}$ (44) associated with these circular modes describe the kinetic momenta

$$k_y = p_y + x/\beta_s, \quad k_x = p_x - y/\beta_s \quad (56)$$

and coordinates of the Larmor center

$$d_x = x/2 - \beta_s p_y/2, \quad d_y = y/2 + \beta_s p_x/2; \quad (57)$$

namely,

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} = \sqrt{\frac{\beta_s}{2}} \begin{pmatrix} k_y \\ k_x \end{pmatrix}, \quad (58)$$

$$\begin{pmatrix} \tilde{a}_3 \\ \tilde{a}_4 \end{pmatrix} = -\sqrt{\frac{2}{\beta_s}} \begin{pmatrix} d_x \\ d_y \end{pmatrix}.$$

These special canonical coordinates in the solenoidal field were proposed in Ref. [21]; they are considered in Ref. [6] in more detail as cyclotron and drift canonical variables.

Let the planar-to-circular adaptive transformation \mathcal{C} be matched with an adjacent downstream solenoid, i.e., $\alpha = 0, \beta = \beta_s$. In this case, the horizontal degree of freedom of the incoming uncoupled beam transforms into the cyclotron mode inside the solenoid, while the vertical one transforms into the drift mode. Due to symplecticity, the corresponding emittances are equal:

$$\begin{aligned} \varepsilon_x^2 &\equiv \langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2 = \varepsilon_c^2 \equiv \langle \tilde{a}_1^2 \rangle \langle \tilde{a}_2^2 \rangle - \langle \tilde{a}_1 \tilde{a}_2 \rangle^2 \\ &= (4/\beta_s^2) (\langle k_x^2 \rangle \langle k_y^2 \rangle - \langle k_x k_y \rangle^2), \\ \varepsilon_y^2 &\equiv \langle y^2 \rangle \langle p_y^2 \rangle - \langle y p_y \rangle^2 = \varepsilon_d^2 \equiv \langle \tilde{a}_3^2 \rangle \langle \tilde{a}_4^2 \rangle - \langle \tilde{a}_3 \tilde{a}_4 \rangle^2 \\ &= (\beta_s^2/4) (\langle d_x^2 \rangle \langle d_y^2 \rangle - \langle d_x d_y \rangle^2), \end{aligned} \quad (59)$$

with the brackets $\langle \dots \rangle$ standing for an ensemble averaging. For a particular case of the round beam inside the solenoid, when $\langle d_x^2 \rangle = \langle d_y^2 \rangle \equiv d^2$, $\langle d_x d_y \rangle = 0$ and similar momentum relations, it yields

$$\varepsilon_x = \beta_s k^2/2, \quad \varepsilon_y = 2d^2/\beta_s. \quad (60)$$

Note that the solenoid with an opposite field switches mapping: the horizontal degree of freedom is mapped onto the drift mode and the vertical plane is mapped onto the cyclotron mode.

Similar relations take place for the reverse, circular-to-planar transformations $\tilde{\mathcal{C}}$.

VII. LOCAL ROTATION INVARIANCE

In a case when the rotation invariance is local (continuous), the circular Courant-Snyder parameters and phases satisfy certain differential equations, similar to the uncoupled case. The derivatives of the β function and phases can be found similar to that in Ref. [5]. For any circular basis vector, the slope $x'(s) = dx(s)/ds$ can be expressed by means of the kinetic momentum: $x' = k_x/p_0 = (p_x - y/\beta_s)/p_0$, with a consequent substitution of the canonical momentum p_x and the coordinate y in terms of the Courant-Snyder parameters

of this basis vector. On the other hand, the slope can be found by a direct derivation of the offset $x(s)$ expressed through the β function and phase. Equating these two expressions for the same value leads to the following relations for the circular modes:

$$\frac{d\beta}{ds} = -\frac{2\alpha}{p_0}, \quad \frac{d\phi_{\pm}}{ds} = \frac{1}{p_0} \left(\frac{1}{\beta} \pm \frac{1}{\beta_s} \right). \quad (61)$$

A size of an axisymmetric laminar beam $r_m(s)$ satisfies the envelope equation [see e.g. [37], Eq. (4.79)]:

$$r_m'' + \frac{\gamma_0' r_m'}{\beta_0^2 \gamma_0} + \frac{\gamma_0'' r_m}{2\beta_0^2 \gamma_0} + \frac{r_m}{\beta_s^2 p_0^2} - \frac{M_m^2}{p_0^2} \frac{1}{r_m^3} - \frac{K}{r_m} = 0. \quad (62)$$

Here β_0 and γ_0 are the relativistic factors, $p_0 = mc\beta_0\gamma_0$ is the total (longitudinal) momentum, M_m is the CAM of the boundary particle with the offset r_m and $K = 2Ie/mc^3\beta_0^3\gamma_0^3$ is the so-called generalized perveance, which takes into account the space charge. The term $\propto \gamma_0'$ gives the adiabatic damping during acceleration, and the term $\propto \gamma_0''$ relates to the electrostatic focusing. The envelope equation gives a simple way to obtain the second-order equation for the circular β function. Indeed, the laminar beam is a beam where only one of the two circular modes is excited; thus, according to Eq. (42), $r_m = \sqrt{\beta|M_m|}$, which leads to an equation for the circular β function:

$$\beta'' - \frac{\beta'^2}{2\beta} + \frac{(\gamma_0'\beta)'}{\beta_0^2 \gamma_0} + \frac{2\beta}{p_0^2} \left(\frac{1}{\beta_s^2} - \frac{1}{\beta^2} \right) - \frac{2K}{|M_m|} = 0. \quad (63)$$

VIII. DIAGONALIZATION OF BEAM MATRIX

Beam distributions are conventionally described by means of the so-called Σ matrix, or the matrix of second moments, $\Sigma \equiv \langle \mathbf{x} \otimes \mathbf{x} \rangle$, with the sign \otimes standing for the outer product and \mathbf{x} for the 4D phase-space vector (1); in other words, $\Sigma_{i,j} = \langle \mathbf{x}_i \mathbf{x}_j \rangle$ (see, e.g., Ref. [2], p. 56). If \mathcal{M} is an arbitrary 4×4 transfer matrix, then the resulting new Σ matrix is determined by $\mathcal{M}\Sigma\mathcal{M}^T$. Unimodular transformations preserve a determinant of the Σ matrix. This relates to all the symplectic transformations, but not only: a transfer from the kinetic to the canonical momenta is not symplectic, but its determinant is also a unit; thus the Σ -matrix determinant is the same in the kinetic and canonical bases. The square root of this determinant is the beam emittance in the 4D phase space. The uncoupled state is described by the block-diagonal Σ matrix in the original Cartesian coordinates (1); its 4D emittance is a product of the 2D emittances. Normally the phase distributions are homogeneous in this case the Σ matrix is diagonal in the matched planar basis (the transfer matrix in this case $\mathcal{M} = \mathbf{V}^{-1}$)

$$\Sigma = \text{diag}(\varepsilon_x, \varepsilon_x, \varepsilon_y, \varepsilon_y), \quad (64)$$

where $\text{diag}(\dots)$ is a diagonal matrix with elements listed as the arguments. Suppose this uncoupled beam is transformed into a round beam by the planar-circular adapter. In the

matched circular basis, this new vortex state has the same diagonal beam matrix (64). However, this vortex state represents an arbitrary round beam; thus, it can be concluded that the Σ matrix of any round beam distribution is diagonalized in a proper circular basis. Assuming that the matrix of a round beam is given in the original Cartesian coordinates \mathbf{x} , its two pairs of circular eigenvectors and two canonical emittances can be found; this is a problem treated in this section.

First of all, the Σ matrix of a round beam can be expressed in rotation-invariant terms. Substitution of

$$\begin{aligned} x &= r \cos \theta, y = r \sin \theta, \\ p_x &= p_n \cos \theta - p_t \sin \theta, \\ p_y &= p_n \sin \theta + p_t \cos \theta, \\ p_x^2 + p_y^2 &= p_n^2 + p_t^2 \equiv p^2 \end{aligned} \quad (65)$$

and averaging over the angle θ leads to the following 2×2 block form of the 4×4 Σ matrix:

$$\begin{aligned} \Sigma &= \frac{1}{2} \begin{pmatrix} \Sigma & \langle r p_t \rangle \mathbf{J} \\ -\langle r p_t \rangle \mathbf{J} & \Sigma \end{pmatrix}, \\ \Sigma &= \begin{pmatrix} \langle r^2 \rangle & \langle r p_n \rangle \\ \langle r p_n \rangle & \langle p^2 \rangle \end{pmatrix}. \end{aligned} \quad (66)$$

Here, the 2×2 matrix \mathbf{J} is determined in Eq. (13); the normal and tangential canonical momenta p_n, p_t are independent of the angle θ due to the beam symmetry.

It is straightforward to check that this beam matrix is diagonalized by the circular basis (35) with

$$\begin{aligned} \beta &= \frac{\langle r^2 \rangle}{\sqrt{\langle r^2 \rangle \langle p^2 \rangle - \langle r p_n \rangle^2}}, \\ \alpha &= -\frac{\langle r p_n \rangle}{\sqrt{\langle r^2 \rangle \langle p^2 \rangle - \langle r p_n \rangle^2}}, \end{aligned} \quad (67)$$

and arbitrary phases ϕ_+, ϕ_- . In this basis, the beam matrix (66) is presented as

$$\Sigma = \text{diag}(\varepsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_2) \quad (68)$$

with the emittances

$$2\varepsilon_{1,2} = \pm \langle r p_t \rangle + \sqrt{\langle r^2 \rangle \langle p^2 \rangle - \langle r p_n \rangle^2} \geq 0. \quad (69)$$

Note that these partial emittances are preserved by any symplectic transformation: when the beam matrix for a new state is diagonalized, it will have the same form as Eq. (68) with the same eigenvalues $\varepsilon_{1,2}$ as the initial state.

The total 4D emittance is a product of these partial emittances:

$$4\varepsilon \equiv 4\varepsilon_1\varepsilon_2 = \langle r^2 \rangle \langle p_n^2 \rangle + \langle r^2 \rangle \langle p_t^2 \rangle - \langle r p_n \rangle^2 - \langle r p_t \rangle^2. \quad (70)$$

All these results can be expressed in terms of the kinetic momenta $k_{x,y}$, related to canonical ones by Eqs. (56). In terms of the normal and tangential components, this can be presented as

$$p_n = k_n, \quad p_t = k_t - r/\beta_s, \quad (71)$$

which leads to

$$2\varepsilon_{1,2} = \pm (\langle rk_t \rangle - \langle r^2 \rangle / \beta_s) + \sqrt{\langle r^2 \rangle \langle k^2 \rangle - \langle rk_n \rangle^2 - 2\langle r^2 \rangle \langle rk_t \rangle / \beta_s + \langle r^2 \rangle^2 / \beta_s^2} \quad (72)$$

and

$$4\epsilon = \langle r^2 \rangle \langle k_n^2 \rangle + \langle r^2 \rangle \langle k_t^2 \rangle - \langle rk_n \rangle^2 - \langle rk_t \rangle^2; \quad (73)$$

the last result was previously found in Ref. [22]. Note that presentations of the 4D emittance in terms of the canonical and kinetic momenta are absolutely identical: a transfer from one to another is equivalent to rotation imposed on the beam as a whole, which does not change the total emittance (70).

The basis which makes the beam matrix diagonal can be considered as eigenvectors of the given beam distribution, while the partial emittances can be seen as eigenvalues. The deduced circular eigenvectors for a round beam give a solution to a problem of its transformation into an uncoupled beam, when the Σ matrix of the round beam is known. Indeed, a circular-to-planar adapter with the circular parameters (67) would complete this job.

As an example, a beam born at the magnetized cathode can be considered. At the round cathode, $\langle r^2 \rangle \equiv 2\sigma_c^2$, $\langle k^2 \rangle \equiv 2mT_c$, $\langle rk_n \rangle = \langle rk_t \rangle = 0$, where σ_c is its rms size and T_c is the temperature; for a homogeneous circle of radius a_c , the rms size $\sigma_c = a_c/2$. The circular Courant-Snyder parameters for the eigenvectors (67) come out as

$$\alpha = 0, \quad 1/\beta^2 = 1/\beta_s^2 + k_T^2/\sigma_c^2, \quad (74)$$

and the emittances [5]

$$\varepsilon_{1,2} = (\sigma_c^2/\beta_s) \sqrt{1 + \beta_s^2 k_T^2 / \sigma_c^2 \mp 1}, \quad \epsilon \equiv \varepsilon_1 \varepsilon_2 = \sigma_c^2 m T_c. \quad (75)$$

If the beam is strongly magnetized, $\beta_s \ll \sigma_c k_T$, then

$$\varepsilon_1 = \beta_s k_T^2 / 2, \quad \varepsilon_2 = 2\sigma_c^2 / \beta_s. \quad (76)$$

A problem of eigenvectors for arbitrary (nonround, coupled) beam distribution has been recently solved by Lebedev and Bogacz [5]. It has been found that the beam matrix can be diagonalized, and the two emittances $\varepsilon_{1,2}$ are given by the positive roots of a characteristic equation for ε :

$$\det[\Sigma^{-1} - (i/\varepsilon)\mathcal{S}] = 0,$$

where $i = \sqrt{-1}$.

IX. POSSIBLE APPLICATIONS

In this section, a possible use of the circular mode formalism and beam adapters is discussed.

A. Round beams for circular colliders

For circular colliders, it should be beneficial to have round beams in the interaction point (IP); a list of references can be found in Ref. [9]. The main reason is that the rotation symmetry of a kick from the round opposite beam accompanied by the revolution matrix invariance leads to angular momentum preservation. This makes the transverse motion equivalent to one dimension. Resulting elimination of the betatron resonances is of crucial importance since they are believed to cause the beam lifetime degradation. Optical realization of the round colliding beams has been proposed in Ref. [8], and a similar scheme has been implemented at CESR [20]. For all these cases, the identity of the horizontal and vertical emittances and tunes is required. Another approach to get the beams round, the Möbius accelerator [23], based on beam rotator optics [24], is studied experimentally at CESR [25]. This scheme also leads to emittance identity and effective tune degeneration: the resulting normal tunes are inevitably separated by 1/2. Use of the matched adapter at the IP opens a way that is free from all these limitations. The matched planar-to-circular and reverse adapters make the beams round only in the space between these adapters. This ‘‘beam rounder’’ does not change the uncoupled beta functions and emittances in the outer part of the storage ring, which would allow us to use it as a transparent insert at existing circular colliders. Generally speaking, inserting this device would change the tunes, which can be restored by another local insert. The adapter is absolutely indifferent to such global parameters as tunes. Two tunes of the storage ring with the local beam rounder are independent variables, both available for the working point optimization. The colliding beams are round for any emittance ratio, and the revolution matrix for any point between the two adapters is rotation invariant. All this guarantees the angular momentum preservation at the beam-beam collisions. Note that the beams would be round not only in the IP itself, but at the whole interval including IP and bounded by the nearest upstream and downstream quadrupoles. A solenoidal magnetic field in the interaction region is not important for the CAM preservation; thus the adapter can be used either with or without the solenoid inside.

B. Flat electron beams for linear colliders

The magnetized-to-flat transformation was suggested to be used for preparation of flat electron beams for linear colliders [14], as an alternative to flat beams obtained in damping rings. This method also allows us to form electron beams with optimum density distribution in the beam plane to obtain maximum luminosity of a collider. The magnetized-to-flat transformation maps the cathode shape onto, say, a horizontal phase space of the outgoing flat beam. Changing the cathode shape, the surface density distribution of the flat beam can be arbitrarily modified, so any distribution function

can be prepared. This optimization would be different for an e^-e^+ collider and e^-e^- collider, since the positron beam is shaped in a damping ring.

The magnetized-to-flat transformation was also suggested to be used for production of Smith-Purcell radiation [26] and intense femtosecond x-ray pulses [27].

C. Relativistic electron cooling

Several beam optics advancements can play a critical role in the development of the relativistic electron cooling projects for hadron beams [6,12,28].

At Fermilab, a project is being developed for electron cooling of antiprotons in the Recycler storage ring at 8.9 GeV/c [29]. To provide beam focusing, the cooling section has to be immersed in the solenoidal field. To avoid beam excitation in the cooling section, the cathode, where the beam is born, has to be properly magnetized, providing the same magnetic flux through the beam as in the cooling section. It is important that all the rest of the transport line be free of extended solenoids [6]. At DESY, a possibility is studied for an RF linac-based electron cooling of 20 GeV protons in PETRA [30]. For both of these projects, the electron beam is CAM-dominated; similar optical problems have to be solved and the same methods can be used.

For high-energy electron cooling, with the energy per nucleon ≥ 100 GeV, the electron beam can be circulating in a storage ring. The effects of intrabeam scattering for this beam are minimized, if it is flat for the greater part of the ring, as it is naturally for an uncoupled lattice. A calm and round beam in the magnetized cooling section can be provided by means of the adapting optics. Schemes of this kind were proposed for Tevatron [32] and RHIC [31] at full energy.

Recent successes in the realization of the energy recovery principle in superconducting electron linacs [33] opens a very promising perspective of linac-based high energy electron cooling. Currently, there are two proposals of this type under development: cooling of heavy ion and proton beams in RHIC [34], and ion cooling in an electron-ion collider [35]; both are based on principles of electron beam transport with a discontinuous solenoid [6]. In view of a high value of electron (average) current required for efficient electron cooling, the incorporation of an electron recirculator ring with the electron linac seems to be an important advancement for future electron cooling devices [36]. Today, this possibility is realized conceptually as a ring with circular modes matched with a solenoid of the cooling section [12,28,38]. In order to extend the lifetime of a high quality beam against intrabeam scattering and (or) quantum radiation, the ring lattice can be complemented by adapters to keep the beam flat in arcs (similar to the above-mentioned electron storage ring case, although the wigglers are not needed here).

There is an interesting possibility to compensate the optical coupling, introduced by the cooling solenoid to the hadron beam. It can be done by changing the sign of the electron CAM in the middle of the cooling section, where the solenoid is disrupted for a special short part of the trajectory, as

discussed in Sec. IV. Assuming this CAM flip is provided, the beam enters the second solenoid, where the magnetic field is reversed, so the beam remains calm there as well as at the first solenoid. This CAM-flip transformation can be provided by two adapters: the first one transforms the CAM-dominated beam into a flat beam, and the second transforms this flat beam into a whirled beam again with an opposite sign of the CAM. As a result of this trick, an average value of the magnetic field in the cooling section is eliminated, which can be beneficial for the cooled particles. This CAM flip requires 5 quadrupoles: adjacent quadrupoles of two skew triplets can be merged.

The transport of the magnetized electron beam from the electron sources to the cooling section at high energies would also make efficient the electron cooling of high energy positron beams [28]. Due to small positron mass and magnetization, this process is very intense [39,40]; employing sweeping and rate-redistribution dispersive techniques [12,41] could additionally intensify it. The circulating positron beam can be cooled down by a linear electron beam to the emittance of a much lower value than that of the electron beam, obtained from a magnetized source. Finally, the cooled positron beams can be used, in their turn, for a fast and deep cooling of circulating electron beams.

D. Low energy hadron cooler rings with circular modes

The equilibrium emittances of a beam under cooling can be limited by Coulomb repulsion when cooling intense low-energy beams. The above-mentioned concept of round beams in a recirculator ring with circular modes matched with a solenoid of the cooling section prompts a possible way to reduce the space charge effect on the 4D phase space emittance [12]. The principal optical feature of such a ring is that the drift and cyclotron components of the hadron particle motion in the solenoid are not mixed by the outside optical channel. Then, the cyclotron component (related to the hadron beam temperature in the solenoid) will experience a deep cooling, not limited by the space charge. The drift component (i.e., beam size) can be cooled to an equilibrium limited by the space charge, using the dispersive cooling.

E. Ionization cooling

A central problem for muon colliders and neutrino factories is the effective ionization cooling of muons. When the muons are transported inside an extended solenoid, only their cyclotron mode, related to the Larmor rotation, can be cooled, while the drift emittance, related to positions of the Larmor centers, is preserved. To make the cooling process comprehensive, a transport scheme with an alternating sign of the magnetic field was proposed. An optimized scheme based on the use of long solenoids was suggested in Ref. [42]. The central idea of this proposal is a cross mapping of the drift and cyclotron modes for the sequential reversed solenoids. Due to the rotation invariance of this symplectic transformation, it could be done by round lenses (short solenoids) or invariant blocks [6]. The canonical angular momentum is preserved by these optics, while cooling makes the CAM value systematically decrease.

F. Electron and ion beams for applied use

High quality relativistic electron beams obtained from magnetized sources can be used for the effective generation of hard radiation, coherent or incoherent. The electron beam in the generation section can be returned (reinject) into a solenoid (a strong one), to eliminate beam rotation. Optionally, the beam can be turned to the flat one, to obtain maximum electron concentration, if necessary. The flat electron and ion beams also might be of interest for technological applications. Flat ion beams with low 4D emittance can be obtained by means of electron cooling in ion rings with circular modes, as described in Sec. IX D. After cooling, a round ion beam can be transformed into a flat one by using a beam adapter in the regime of circulation or after ejection from the ring. Optionally, if the use of a very cold ion beam would be compatible with the magnetic field, the beam can be reinjected into the solenoid of the user section, keeping it in a tranquil round state.

X. SUMMARY

In the optics of charged particle beams, circular transverse modes can be introduced; they might be considered as analogous to the circular modes in the optics of light. These modes provide an adequate basis when the transformations are rotation invariant. A group of the invariant transformations is shown to be identical to a group of transformations preserving the canonical angular momentum; its matrices are described. The rotation-invariant mappings and circular modes can be parametrized in a way which makes them similar to

the Courant-Snyder parametrization in the conventional uncoupled case. The constructed symplectic basis of circular modes make almost obvious an idea of the beam adapters, which are optical transformers of the planar to circular modes and back. The adapters can be implemented on a base of a skew quadrupole block; mapping of a given planar basis onto a given circular basis requires six quadrupoles with variable field gradients. Inside a solenoid, there is a particular choice for the circular modes, when one of them describes the cyclotron rotation and another describes coordinates of the Larmor center. In the case of a beam born at the magnetized cathode, another special choice of the circular modes allows us to present a matrix of the beam second moments (the so-called Σ matrix) in a diagonal form. A proper downstream adapter can transform this beam into an uncoupled, or an $X - Y$ -uncorrelated state, in which horizontal and vertical emittances are equal to the corresponding circular emittances of the beam at the cathode. Such transformations can be used for flat beams preparation in linear colliders. Beam adapters can also be used for preparation of round beams in the interaction region of the circular colliders. Requiring only local matching and being insensible to the machine tunes, the beam adapters can be added without any change to the main part of the lattice. Providing round beams and a rotation-invariant revolution matrix, such inserts guarantee the angular momentum preservation, which is believed to be crucial for a significant increase of luminosity. Relativistic electron cooling of heavy particles and ionization cooling of muons present other fields of research where use of the circular modes can be quite relevant.

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